A Classical Linear λ-Calculus based on Contraposition

joint work with **Pablo Barenbaum** and **Eduardo Bonelli**

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Redexes in λ_{MLL}.

We have the following three redexes:

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The following rule is admissible:

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Reduction and Structural equivalence

Let $\mathsf{L}, \mathsf{L}', \ldots$ stand for lists of $\otimes$-eliminators: $\quad \mathsf{L} ::= \Box \mid \mathsf{L}[\langle a, b\rangle := t]$.

Reduction rules (at a distance; cf. Accattoli & Kesner, 2010)

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t[\langle a, b\rangle := \langle s, r\rangle \mathsf{L}] & \to & t\{a := s\}\{b := r\}\mathsf{L} \\
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Structural equivalence: We define the equivalence $\equiv$ that allows $\otimes$-eliminators to "float":

$$\mathsf{C}\langle \ldots t[\langle a, b\rangle := s] \ldots\rangle \equiv \mathsf{C}\langle \ldots t \ldots\rangle[\langle a, b\rangle := s]$$

Properties of the calculus.

- **Logical soundness/completeness**.
  $\vdash \Gamma, A$ is valid in MLL iff there is a term t such that $\Gamma^{\perp} \vdash t : A$.
- **Subject reduction**.
  If $t \to t'$ and $\Gamma \vdash t : A$ then $\Gamma \vdash t' : A$.

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  If $t \twoheadrightarrow t_1$ and $t \twoheadrightarrow t_2$ then there are t_1' and t_2' such that: $t_1 \twoheadrightarrow t_1'$, $t_2 \twoheadrightarrow t_2'$ and $t_1' \equiv t_2'$.
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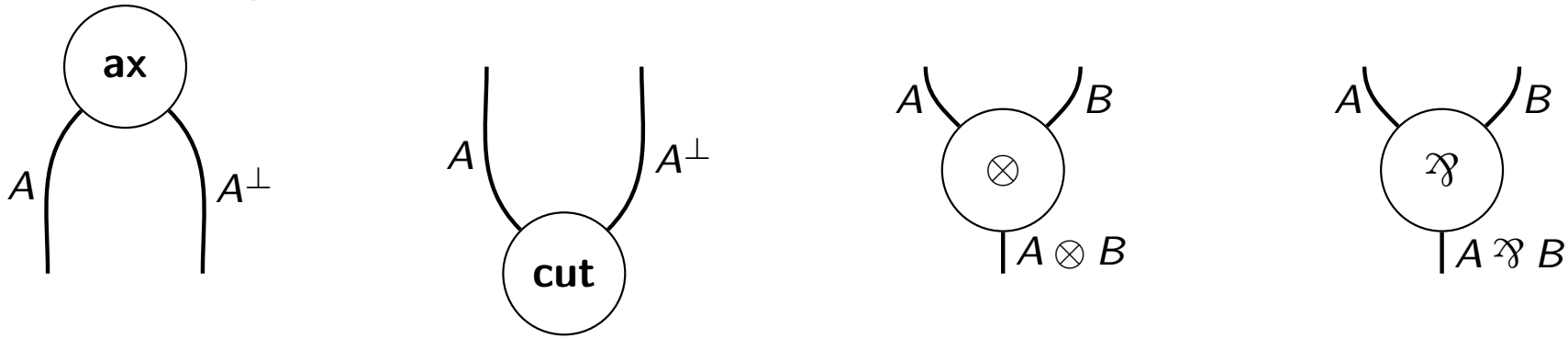
Proof nets for MLL

Proof nets are finite acyclic oriented graphs built over the alphabet of the following nodes:



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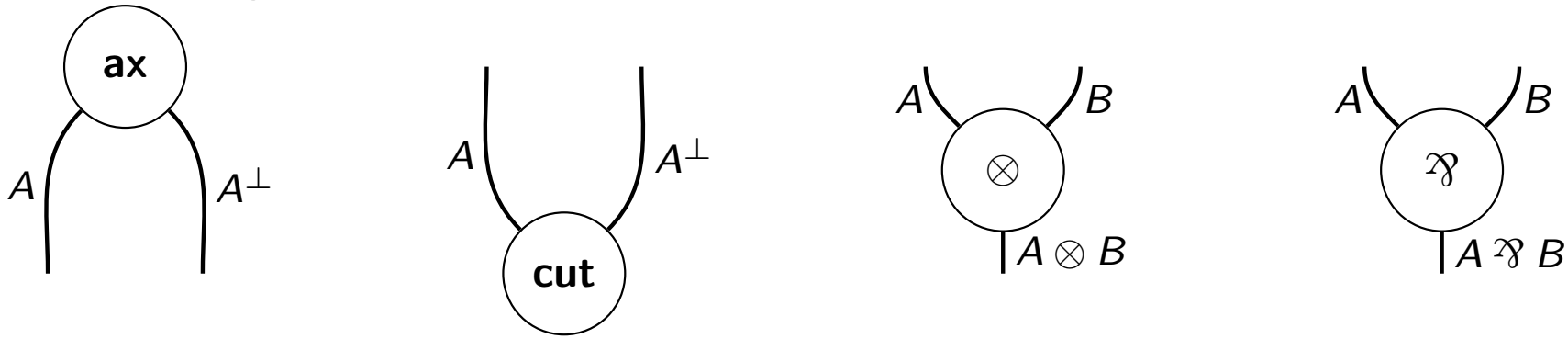

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- The incident (top) edges are **premises**.

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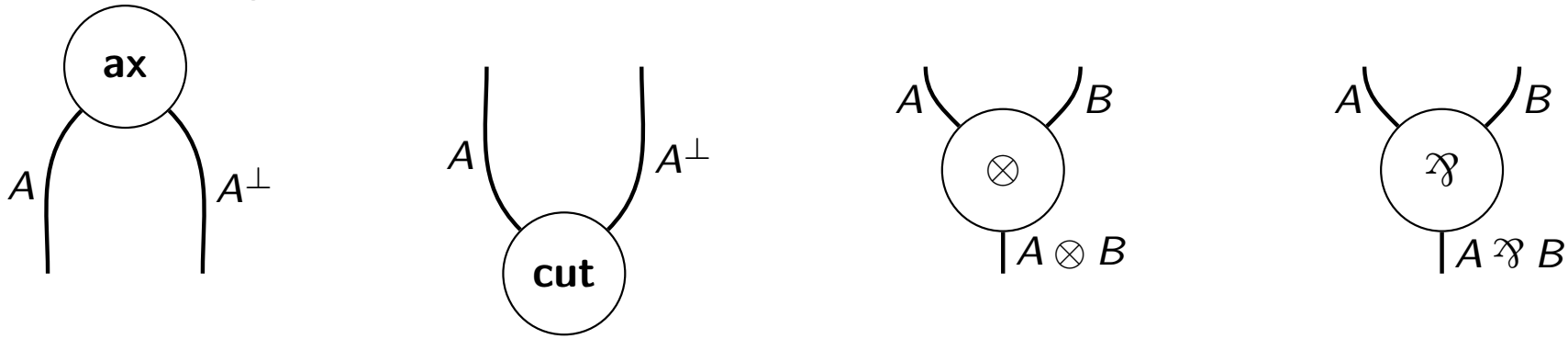
Proof nets are finite acyclic oriented graphs built over the alphabet of the following nodes:



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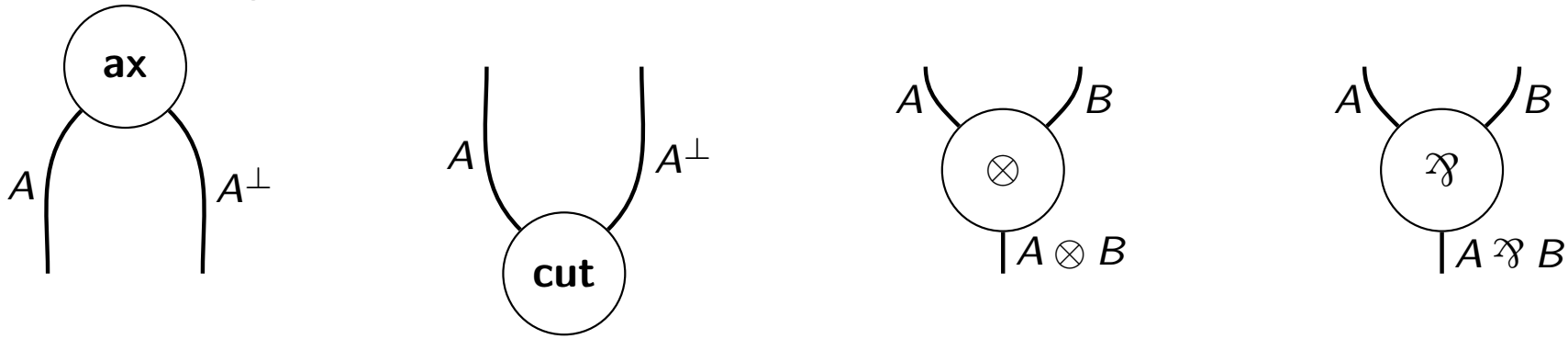

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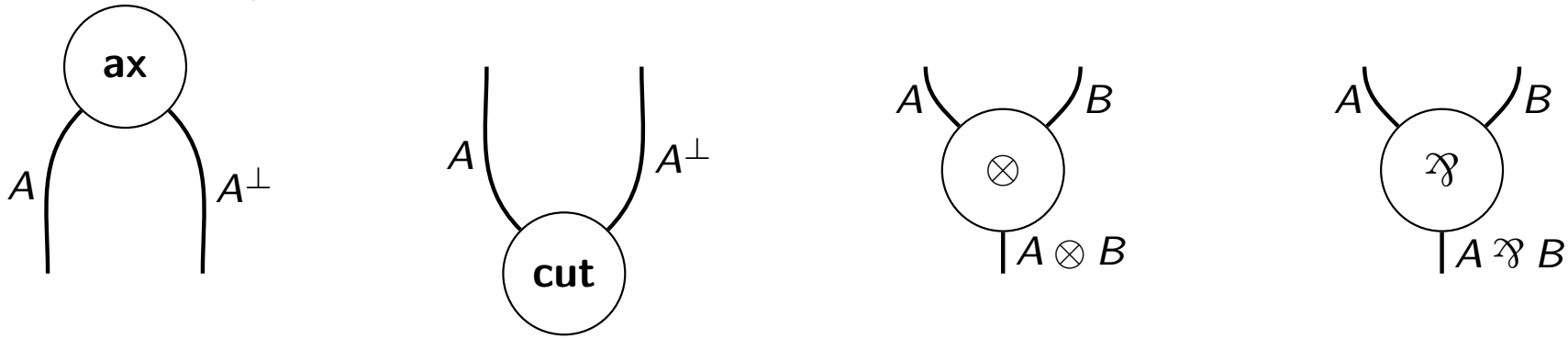

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The proof nets come equipped with a cut elimination reduction for ax, $\parr/\otimes$.

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Following Laurent (2003) we will distinguish one of the conclusions of a proof net. Graphically, the distinguished conclusion will be denoted by $\rightarrow$, while the old distinguished conclusions will be denoted by $\multimap$.

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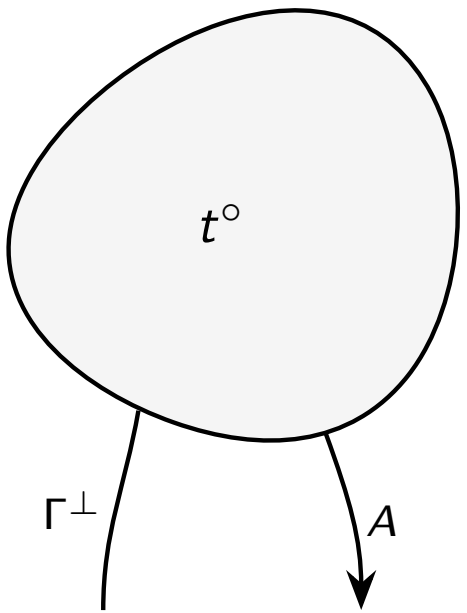
We will map terms from the calculus λ_{MLL} to a proof net with a distinguished conclusion

$$\Gamma \vdash t : A \qquad \overset{\circ}{\longmapsto}$$



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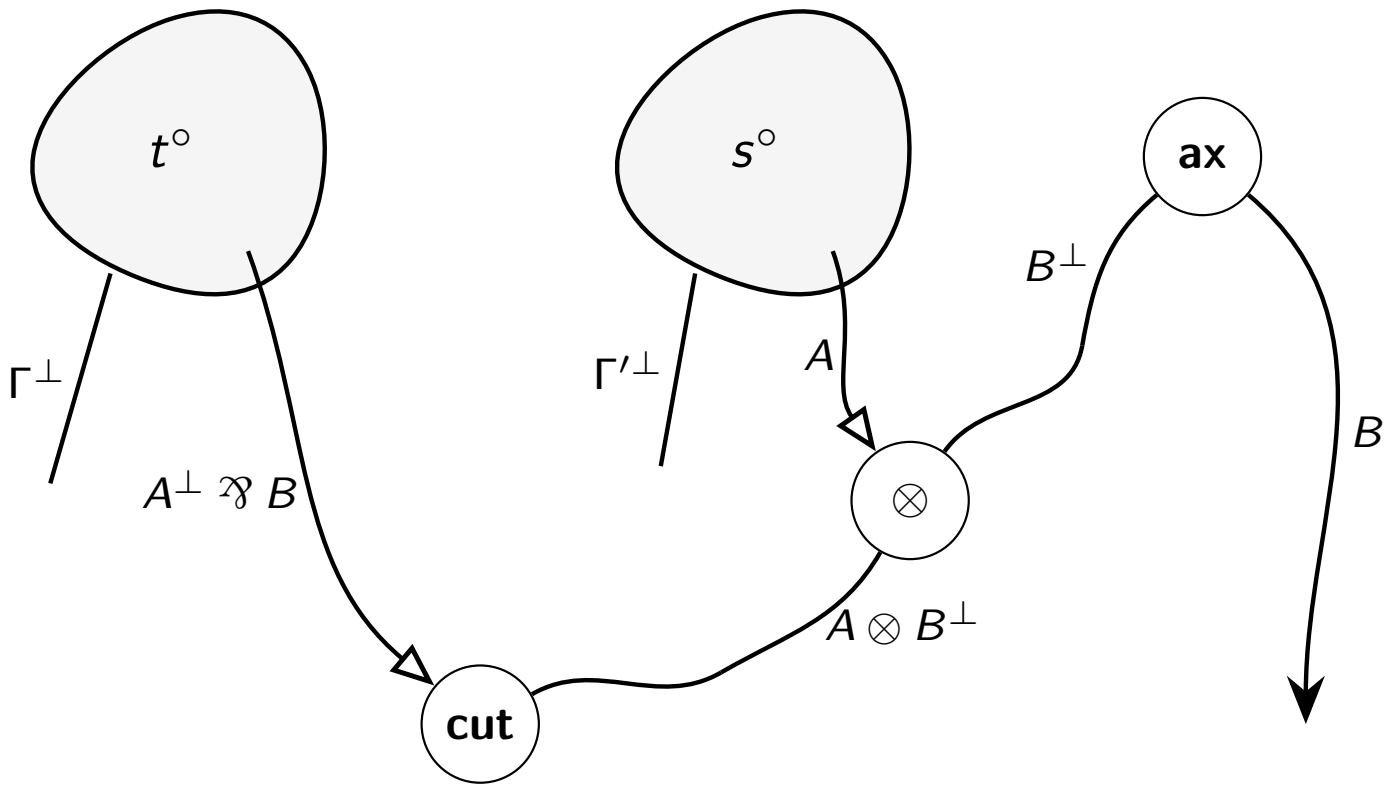
$$\Big\Vert$$



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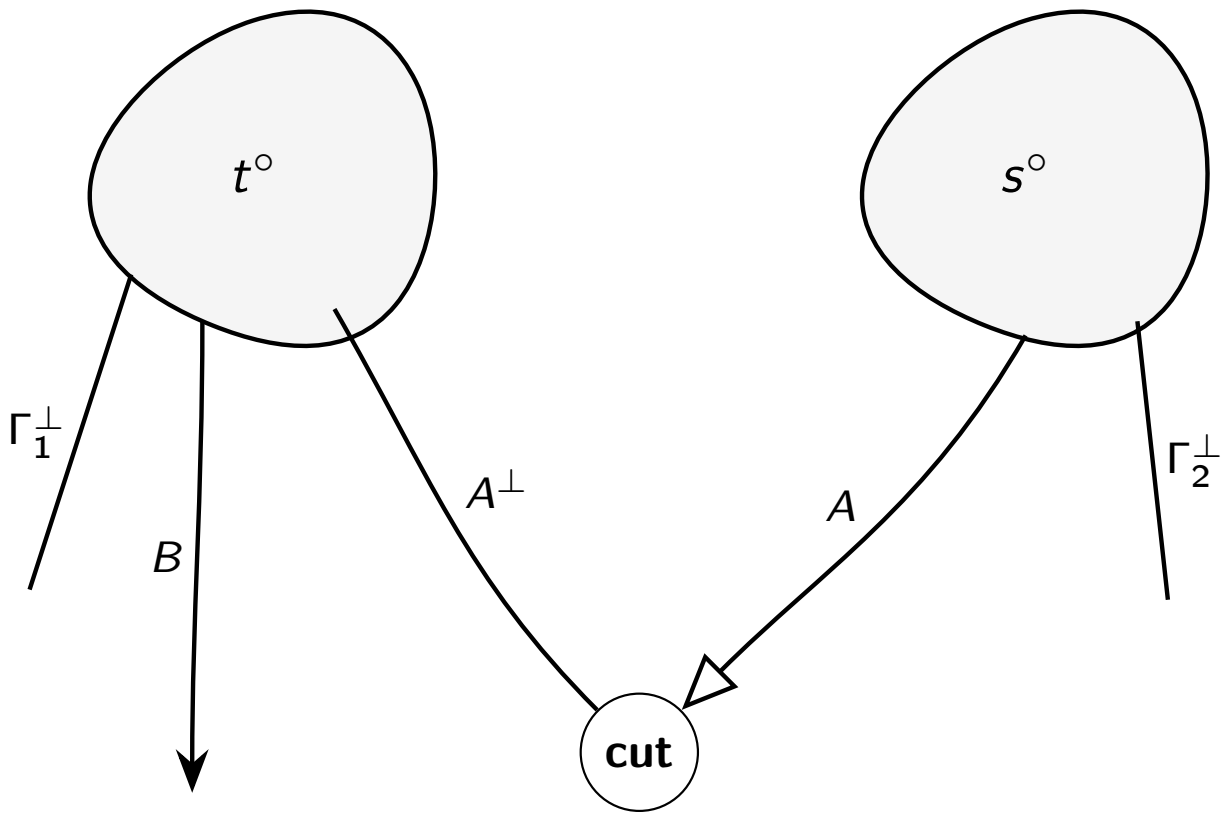
Substitution lemma

$$\frac{\Gamma_1, a : A \vdash t : B \quad \Gamma_2 \vdash s : A}{\Gamma_1, \Gamma_2 \vdash t\{a := s\} : B}\ \text{sub}$$

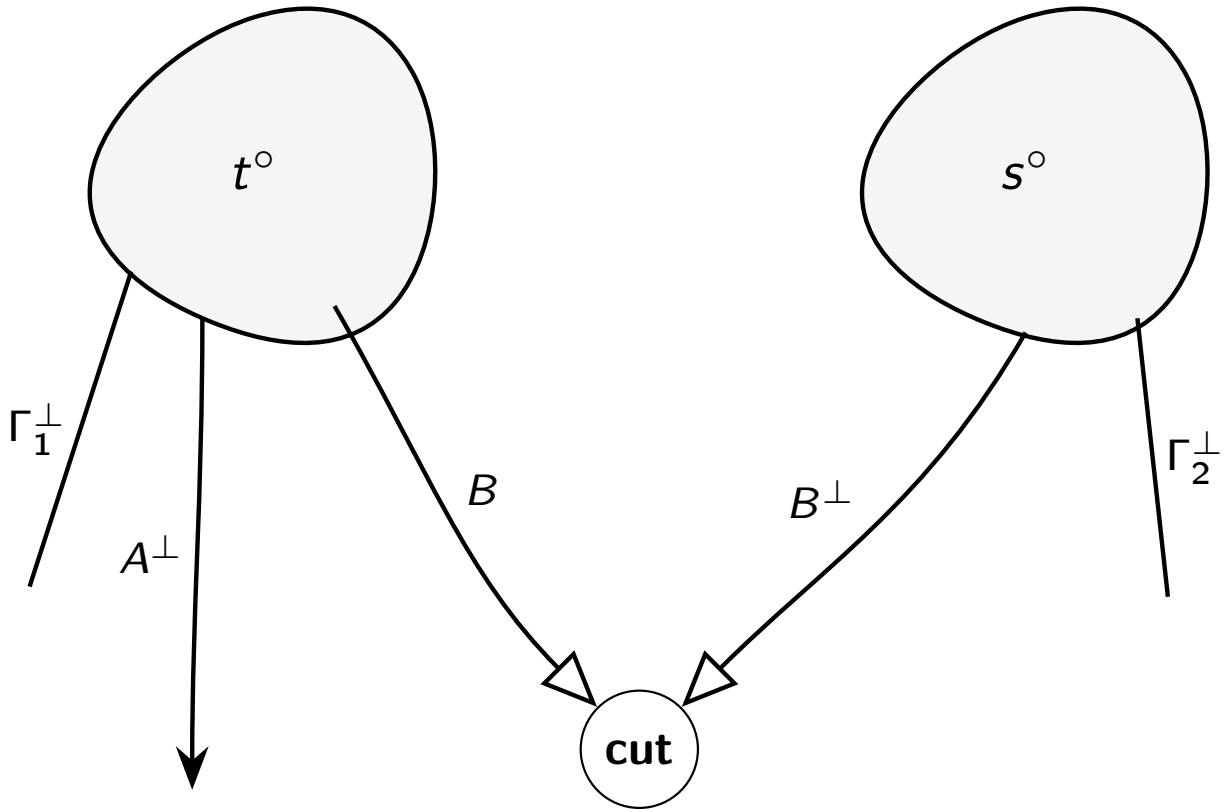


$\twoheadrightarrow (t\{a := s\})^\circ$

Contrasubstitution

Contrasubstitution lemma

$$\frac{\Gamma_1, a : A \vdash t : B \quad \Gamma_2 \vdash s : B^\perp}{\Gamma_1, \Gamma_2 \vdash t\{a \backslash\!\backslash s\} : A^\perp}\ \text{contra}$$



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Syntax for $\lambda_{\mathbf{MELL}}$

$$A, B, \ldots ::= \alpha \mid \alpha^{\perp} \mid A \otimes B \mid A \parr B \mid \mathbb{1} \mid \perp \mid !A \mid ?A$$

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Typing rules 1/2

λ_{MELL} uses two contexts, as DILL: (Barber, 1996)

- **Unrestricted** contexts $\Delta, \Delta', \ldots$ binding unrestricted variables $u, v, \ldots$.
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$\parr$ typing rules

$$\frac{\Delta; \Gamma, a : A, b : B \vdash t : \bot}{\Delta; \Gamma \vdash \parr(a, b).t : A^{\bot} \parr B^{\bot}}\ \text{m-i}\parr$$

$$\frac{\Delta; \Gamma \vdash t : A \parr B \quad \Delta; \Gamma' \vdash s : A^{\bot}}{\Delta; \Gamma, \Gamma' \vdash t\ s : B}\ \text{m-e}\parr_1 \qquad \frac{\Delta; \Gamma \vdash t : A \parr B \quad \Delta; \Gamma' \vdash s : B^{\bot}}{\Delta; \Gamma, \Gamma' \vdash t \blacktriangledown s : A}\ \text{m-e}\parr_2$$

Typing rules 2/2

Exponential typing rules

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$$\frac{\Delta, u : A^{\bot}; \Gamma \vdash t : \bot}{\Delta; \Gamma \vdash ?u.t : ?A}\ \text{m-i?} \qquad \frac{\Delta; \Gamma \vdash s : ?A \quad \Delta; a : A \vdash t : \bot}{\Delta; \Gamma \vdash t \blacktriangleright^{a} s : \bot}\ \text{m-e?}$$

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Units typing rules

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  but they don't lead to a well-behaved contra-substitution operation.
- If we add the rule (in the style of Barbanera, Berardi (1996))

$$\frac{\Delta; \Gamma, a : A \vdash t : \bot}{\Delta; \Gamma \vdash \lambda a.\, t : A^\bot}$$

  where $\lambda a.\, t$ is shorthand for $\parr(a, b).t \leftlightning b$

  Then we could drop all of m-e$\otimes$, m-e$\parr_1$, m-e$\parr_2$, m-e!, m-e? and m-e$\mathbb{1}$. However, it ends up being non-confluent.

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⚠Unluckily, this doesn't guarantee confluence, thus we are forced to add the following two pairs of symmetrical reduction rules:

$$
\begin{array}{lll}
(\parr(a,b).t)\mathsf{L} \lightning \langle s, r\rangle \mathsf{K} & \mapsto_{\parr\otimes} & t\{a := s\}\{b := r\}\mathsf{LK} \quad (b \notin \mathrm{fv}(s)) \\
\langle s, r\rangle \mathsf{L} \lightning (\parr(a,b).t)\mathsf{K} & \mapsto_{\otimes\parr} & t\{a := s\}\{b := r\}\mathsf{LK} \quad (b \notin \mathrm{fv}(s)) \\
(!a.t)\mathsf{L} \lightning (?u.r)\mathsf{K} & \mapsto_{!?} & r\{u := t\{a \backslash\!\backslash \star\}\}\mathsf{LK} \\
(?u.r)\mathsf{L} \lightning (!a.t)\mathsf{K} & \mapsto_{?!} & r\{u := t\{a \backslash\!\backslash \star\}\}\mathsf{LK}
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We let eliminators float.

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Proofs of $\mathbb{1}$ are interchangeable when they are eliminated

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We let eliminators float.

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- Where S is a context such that its hole is not inside $!a.\Box$ or $\Box\blacktriangleright^a s$

$$t[\star := \star] \quad \equiv_{\beta\star} \quad t$$

The $I\bot/E\bot$ redex is treated as an equation

$$t[\star := s] \quad \equiv_{\mathsf{Symm}\star} \quad s[\star := t]$$

Proofs of $\mathbb{1}$ are interchangeable when they are eliminated

$$t\{a \Vdash \star\}\lightning r \quad \equiv_{\mathsf{split}} \quad t\{a := r\}$$

Every term of type $\bot$ arises as a contradiction

Properties of the calculus

- **Logical soundness / completeness**.
  $\vdash \Gamma, A$ is valid in MELL iff there is a term t such that $\cdot; \Gamma^{\perp} \vdash t : A$.
- **Subject reduction**.
  If $\Delta; \Gamma \vdash t : A$ and $t \to s$ then $\Delta; \Gamma \vdash s : A$.
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- **Confluence modulo** $\equiv$.
  If $t \twoheadrightarrow t_1$ and $t \twoheadrightarrow t_2$ then there are t_1' and t_2' such that: $t_1 \twoheadrightarrow t_1'$, $t_2 \twoheadrightarrow t_2'$ and $t_1' \equiv t_2'$.

The $\lambda\mu$-calculus

It is a calculus for classical logic, first appeared in Parigot (1992).

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$$\begin{array}{rcl} A, B, \ldots & ::= & \bot \mid \alpha \mid A \supset B \\ M, N, \ldots & ::= & x \mid \lambda x.\, M \mid M\, N \mid \mu a.M \mid [a]M \end{array}$$

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Typing rules (for the new constructors)

$$\frac{\Gamma \vdash M : A \mid \Sigma, a : A}{\Gamma \vdash [a]M : \bot \mid \Sigma, a : A}\ \text{p-name} \qquad \frac{\Gamma \vdash M : \bot \mid \Sigma, a : A}{\Gamma \vdash \mu a.M : A \mid \Sigma}\ \text{p-mu}$$

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T-translation for $\lambda\mu$ (formulae) cf. Danos (1995)

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Proposition

- If $\Gamma \vdash M : A \mid \Sigma$ holds in $\lambda\mu$, then $?\Gamma^{\mathsf{T}}, (\Sigma^{\mathsf{T}})^{\bot}, k : (A^{\mathsf{T}})^{\bot}; \cdot \vdash M^{\mathsf{T}} : \bot$ holds in λ_{MELL}
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Conclusion

This work

- New calculi for MLL and MELL based on the construction of the **contrasubstitution**.
- Good properties such as confluence (modulo $\equiv$), strong normalization, etc.
- Translations from classical calculi via T and Q translations cf. Danos (1995):
  - T and Q translation for $\lambda\mu$ Parigot (1992).
  - T translation for $\overline{\lambda}\mu\tilde{\mu}$ Curien-Herbelin (2000).
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Future work

- Relate λ_{MELL} with proof nets.
- Extensions: additives, fixed points, etc
- Can we formulate an untyped version of λ_{MELL}?
- Translations *into* other calculi.

Contrasubstitution – Definition in λ_{MELL}

A term t is **linear** in λ_{MELL} if it is in the λ_{MLL} sense and moreover. 1) subterms of the form $!a.t$ have no free linear variables; 2) and in subterms of the form $t \blacktriangleright^a s$, only a occurs as a free linear variable in t.
Typable terms are linear!

We define contrasubstitution by induction on the structure of a linear term t such that $a \in \mathrm{fv}(t)$. We omit the repeated and impossible cases.

$$
\begin{array}{rcl}
(t_1[\langle b,c\rangle := t_2])\{a \backslash\backslash s\} & := & \begin{cases} t_1\{a \backslash\backslash s\}[\langle b,c\rangle := t_2] & \text{if } a \in \mathrm{fv}(t_1), a \notin \{b,c\} \\ t_2\{a \backslash\backslash \parr(b,c).(t_1 \lightning s)\} & \text{if } a \in \mathrm{fv}(t_2) \end{cases} \\
(\parr(b,c).t)\{a \backslash\backslash s\} & := & t\{a \backslash\backslash \star\}[\langle b,c\rangle := s] \\
t_1[!u := t_2]\{a \backslash\backslash s\} & := & \begin{cases} t_1\{a \backslash\backslash s\}[!u := t_2] & \text{if } a \in \mathrm{fv}(t_1) \\ t_2\{a \backslash\backslash ?u.t_1 \lightning s\} & \text{if } a \in \mathrm{fv}(t_2) \end{cases} \\
(?u.t)\{a \backslash\backslash s\} & := & t\{a \backslash\backslash \star\}[!u := s] \\
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\end{array}
$$

Contrasubstitution – Example in $\lambda_{\mathbf{MELL}}$

Let $t = ?u.(\parr(c,d).r \lightning \langle a, e\rangle)$, where r is a linear term with no occurrence of the linear variable a. We compute $t\{a \backslash\!\backslash s\}$ for a linear term s. First, by definition of contra-substitution on ?-introduction:

$$t\{a \backslash\!\backslash s\} = \parr(c,d).r \lightning \langle a, e\rangle\{a \backslash\!\backslash \star\}[!u := s].$$

Because the sole occurrence of a lies in the right subterm of $\lightning$, this rewrites to $\langle a, e\rangle\{a \backslash\!\backslash \parr(c,d).r\}[!u := s][\star := \star]$. Finally, since a occurs in the left position of the pair, we obtain:

$$a\{a \backslash\!\backslash ((\parr(c,d).r)\blacktriangledown e)\}[!u := s][\star := \star] \;=\; ((\parr(c,d).r)\blacktriangledown e)[!u := s][\star := \star].$$

Need of structural substitution for Confluence – Example in $\lambda_{\mathbf{MELL}}$

Let $t = \overline{(\parr(a,b).\underline{(\parr(c,d).s)[\star := a]\ p)} \blacktriangledown r}$, where, say, $b \in \mathrm{fv}(s)$.